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Non-relativistic fermions interacting through the Chern–Simons field and the Aharonov–Bohm scattering amplitude

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Abstract. Starting from the non-relativistic field theory of spin-$\frac{1}{2}$ fermions interacting through the Abelian Chern–Simons term, we show that the quantized field theory leads, in the two-particle sector, to a two-particle Aharonov–Bohm-like Schrödinger equation with an antisymmetric (fermionic) wavefunction and without a delta function term. Calculating perturbatively the field-theoretic two-particle scattering amplitude up to one-loop order, we show that, in contrast to the scalar theory, the contribution of all the one-loop diagrams is finite and null, and that of the tree level ones coincides with the exact amplitude. Further, the Pauli matter–magnetic field interaction term is shown not to contribute to the amplitude to this order.

1. Introduction

Lately, the Aharonov–Bohm (AB) effect [1] has received again substantial attention in the literature. The main reason behind this revival of interest is its relation to anyons [2], planar particles with fractional spin and statistics. It was shown that such exotic statistics arise when particles of conventional statistics are coupled to the Chern–Simons (CS) gauge field, as this field creates an AB-like interaction that converts the particles to charge-flux-tube composites (see the review [3] and the references therein).

Following this development, a Galilean field theory of scalar fields minimally coupled to the CS gauge field was proposed [4], and shown to lead, in the two-particle sector, to a Schrödinger equation similar to the AB equation.

This recent interest in the AB effect brought back the issue of the calculation of the AB scattering amplitude in the framework of perturbation theory (the Born approximation) which was known to fail for some time [5, 6], and many attempts to propose models for which the Born approximation works were made [7–10]. In particular, the work [10] considered the issue in the framework of Galilean scalar field theory [4], and demonstrated through a perturbative calculation of the two-particle scattering amplitude up to one-loop order, that this amplitude is non-renormalizable, unless a contact interaction is introduced, which for a given strength of the interaction reduces the amplitude to the same order term of the series expansion of the quantum mechanical AB amplitude. The same procedure was generalized later [11] to the non-Abelian case, and similar results were obtained.

This encouraging development led people to address the issue from a more general point of view. For instance, the works [12, 13] raised the interesting question of whether

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the exact (non-perturbative) quantum mechanical AB amplitude can be reproduced order by order perturbatively in the framework of Galilean scalar field theory [4]. The work [12] concludes that the full agreement is obtained, if the renormalized strength of the contact interaction (induced by renormalization) is chosen to be related to the self-adjoint extension parameter, for fixed renormalization scale. The conclusion of the more recent work [13] is not in full aggreement with that of the [12], however. They show that the full agreement can be obtained only in some special regimes. Thus, we see that the general problem in the context of the non-relativistic scalar field theory is not satisfactorily settled yet.

In subsequent works [14], it was shown that if one starts from the relativistic Lagrangian, one finds a renormalizable one-loop scattering amplitude which remains so in the non-relativistic limit as well, thus reproducing the quantum-mechanical result without the need to introduce a contact interaction term. It is not clear yet whether the issue (and the solution) raised in [12, 13] would be the same in the case of relativistic field theories. Obviously, there would be some fundamental differences between the non-relativistic and the relativistic cases. For instance, in the non-relativistic case the necessity of a cut-off is not a relic of some unknown ultraviolet physics, but rather an artefact of the perturbative methods used. This is in contrast with the conventional wisdom on renormalization, whose natural habitat is the relativistic field theories.

A quantum-mechanical treatment of the AB scattering of particles with spin from an infinitely long solenoid was given by different authors [15, 16], and it was shown that the first-order Born approximation in this case works. This problem is considered in this work from a field-theoretic point of view, by considering the field theory of spin-$\frac{1}{2}$ fermions interacting through the CS field.

This paper is organized as follows. In section 2, we introduce our model as the non-relativistic limit of the classical theory of relativistic fermions interacting through the CS gauge field. In section 3, we show that the two-particle sector of the quantum field theory leads to an AB-like equation, and discuss the effect of the fermionic statistics on the scattering amplitude. Section 4 is devoted to the quantum field theory-based perturbative calculations of the two-particle scattering amplitude up to one-loop order, and the demonstration of the role of the fermionic statistics in obtaining a finite, divergence-free amplitude. We sum up our conclusions in section 5.

2. The model

We start with the action of classical relativistic Fermi fields in $2+1$ dimensions coupled to a gauge field whose action is given by the CS term (pure CS field),

$$S = \int \mathrm{d}^3x \left(\bar{\psi}(\mathrm{i}\not{D} - m)\psi + \frac{\mu}{2}\varepsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \right). \tag{1}$$

Here ψ and $\psi^\dagger$, are two-dimensional Grassmann spinor fields,

$$\{\psi, \psi^\dagger\} = \psi\psi^\dagger + \psi^\dagger\psi = 0 \tag{2}$$

$$\not{D} = \gamma_\mu D^\mu \qquad D_\mu = \partial_\mu - \mathrm{i}eA_\mu. \tag{3}$$

The Dirac matrices in $2+1$ dimensions are taken as [17],

$$\gamma_o = \sigma_3 \qquad \gamma_i = \mathrm{i}\sigma_i \qquad i = 1, 2 \tag{4}$$

with the σ being the Pauli spin matrices. These γ-matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{5}$$

$$\gamma_\mu\gamma_\nu = g_{\mu\nu} - \mathrm{i}\varepsilon_{\mu\nu\lambda}\gamma^\lambda \tag{6}$$

the metric $g_{\mu\nu}$ is defined as

$$A_\mu A^\mu = A_\mu g^{\mu\nu} A_\nu \qquad g_{\mu\nu} = \mathrm{diag}(1, -1, -1,) \tag{7}$$

μ and e are two dimensionless coupling constants (we adopt the usual system of units in which $\hbar = c = 1$). A transformation of the gauge field of the form:

$$A_\mu \to \frac{1}{\sqrt{\mu}} A_\mu \qquad e \to \sqrt{\mu} g \tag{8}$$

allows us to have only one coupling constant g, in our theory. Then

$$S = \int \mathrm{d}^3 x \, (\bar{\psi}(\mathrm{i}\not{D} - m)\psi + \tfrac{1}{2}\varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda) \tag{9}$$

with

$$D_\mu = \partial_\mu - \mathrm{i} g A_\mu. \tag{10}$$

The above action, with an additional Maxwell term was first considered by Schonfeld [18] and Deser *et al* [19] where the canonical quantization of this theory was also carried out. The path integral quantization of this theory, and its non-Abelian generalization was recently carried out in [20, 21].

The classical equations follow from varying the action (9) with respect to the fields, and read:

$$(\mathrm{i}\not{D} - m)\psi = 0 \tag{11}$$

$$\varepsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = J_\mu \tag{12}$$

where

$$J_\mu = -g \bar{\psi} \gamma_\mu \psi \qquad \mu = 0, 1, 2. \tag{13}$$

Our aim is to derive the non-relativistic limit of the action (9), so, we write as usual [22]

$$\psi(x, t) = \exp\{-\mathrm{i}mt\} \begin{pmatrix} \phi(\boldsymbol{x}, t) \\ \chi(\boldsymbol{x}, t) \end{pmatrix}. \tag{14}$$

In $2 + 1$ dimensions, $\phi(\boldsymbol{x}, t)$ and $\chi(\boldsymbol{x}, t)$ are single-component spinors. Substituting the above parametrization into equation (11), expressing the lower component of the spinor in terms of the upper one as in the usual routine, and going to the non-relativistic (NR) limit, we reduce the action, equation (9) to

$$S = \int \mathrm{d}^3 x \, \phi^*(\boldsymbol{x}, t) \left(\mathrm{i} D_t + \frac{\boldsymbol{D}^2}{2m} + \frac{gB}{2mc} \right) \phi(\boldsymbol{x}, t)) + \mathcal{S}_{CS} \tag{15}$$

where $B = \boldsymbol{\nabla} \times \boldsymbol{A}$ is the magnetic field. Then, expressing the CS Lagrangian in terms of B also, we find the Lagrangian of our NR model

$$\mathcal{L} = \phi^*(\boldsymbol{x}, t) \left(\mathrm{i} D_t + \frac{\boldsymbol{D}^2}{2m} \right) \phi(\boldsymbol{x}, t) + \frac{g}{2mc} \phi^*(\boldsymbol{x}, t) B \phi(\boldsymbol{x}, t) + A_0 B - \frac{1}{2} \boldsymbol{A} \times \partial_t \boldsymbol{A}. \tag{16}$$

Although as stated before, we are using the universal unit system in which $\hbar = c = 1$, in this work, to make the non-relativistic approximation clearer, we kept the powers of c

explicitly in (15) and (16). Note that, with the c factor put back, $\boldsymbol{D} = \boldsymbol{\nabla} - \mathrm{i}\frac{g}{c}\boldsymbol{A}$. In the foregoing we will suppress them again. Whenever the c factors are to be seen explicitly, one merely replaces g with $\frac{g}{c}$. The Hamiltonian can be constructed in a straightforward manner:

$$H = \int \mathrm{d}^3 x\, \mathcal{H} \tag{17}$$

$$\mathcal{H} = -\phi^* \frac{\boldsymbol{D}^2}{2m}\phi + -\frac{g}{2m}\phi^* B\phi - A_0(B + g\phi^*\phi). \tag{18}$$

The expression in the brackets in the last term in equation (18) above is just the well known analogue of the Gauss' law constraint in electrodynamics, and is the generator of the infinitesimal spatial gauge transformations. A_0 appears here as an arbitrary Lagrange multiplier. One can solve the Gauss' law constraint in a given gauge (in the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, for instance) to express the gauge field in terms of matter fields, obtaining

$$\boldsymbol{A}(\boldsymbol{x}, t) = g\boldsymbol{\nabla} \times \int \mathrm{d}^3 x'\, G(\boldsymbol{x} - \boldsymbol{x}')\phi^*(\boldsymbol{x}', t)\phi(\boldsymbol{x}', t) \tag{19}$$

where $G(\boldsymbol{x} - \boldsymbol{x}')$ is the Green function of the two-dimensional Laplacian,

$$G(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}'|. \tag{20}$$

So, setting the constraint to zero, one obtains the NR Hamiltonian now as

$$\mathcal{H} = -\phi^* \frac{\boldsymbol{D}^2}{2m}\phi - \frac{g}{2m}\phi^* B\phi. \tag{21}$$

Integrating the first term by parts, and expressing the magnetic field B in terms of the matter fields, acoording to (19), yields

$$\mathcal{H} = \frac{1}{2m}(\boldsymbol{D}\phi)^* \cdot (\boldsymbol{D}\phi) + \frac{g^2}{2m}(\phi^*\phi)^2. \tag{22}$$

The quantum theory is now constructed by imposing the equal-time anticommutation relations (since ϕ is a Grassmann field) on the canonically conjugate fields ϕ and ϕ^*,

$$\{\phi(\boldsymbol{x}, t), \phi^*(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{23}$$

A Hamiltonian similar to (22), without the matter–magnetic field coupling term (usually referred to as the Pauli term), and with bosonic statistics was considered by many authors [4, 10]. A Hamiltonian with a term similar to the matter–magnetic field interaction, again with bosonic statistics, that is put 'by hand', was considered first by Jackiw and Pi [23], where the classical field theory was also investigated in addition to the quantum one. In the following section, we are going to investigate the relation of our model Hamiltonian (22) together with the anticommutation relations (23), to the AB scattering problem.

3. The Aharonov–Bohm equation

As we mentioned above, in previous works [4] a scalar field model, similar to our model without the Pauli term, was shown to be a field theoretic formulation of the AB problem, and a Schrödinger equation similar to the AB equation was derived for the two-particle sector of the Hilbert space of this theory. Here, we are going to show that our model also gives a similar equation, with an antisymmetric two-particle wavefunction. However, since we have Fermi statistics in our case, we believe that it would be useful to show in some detail how to derive such an equation since this was not done in details in the literature for theories with fermionic statistics.

In ordering the operators in (22), one should be careful in dealing with non-commuting operators. We adopt a normal ordering prescription in which all the fields ϕ^* appear to the left of ϕ. However, noting the commutator

$$[\phi(\boldsymbol{x}), \boldsymbol{A}(\boldsymbol{y})] = g\boldsymbol{\nabla} \times G(\boldsymbol{y}-\boldsymbol{x})\phi(\boldsymbol{x}) \tag{24}$$

and choosing a prescription in which $\boldsymbol{\nabla} \times G(\boldsymbol{x})$ (which is ill-defined at the origin) vanishes there [23], we see that the ϕ field commutes with the gauge field at the same point, and so does ϕ^*. It can also be shown that the components of the gauge field commute with each other. So, there is no ordering ambiguity afflicting the Hamiltonian. The second term (the Pauli term) in the Hamiltonian (22) vanishes upon normal ordering, since we have products of the same NR fermionic field operators at the same spacetime point.

Now, the Heisenberg equation of motion for ϕ reads:

$$\begin{aligned}
\mathrm{i}\partial_t\phi(\boldsymbol{x}) = [\phi(\boldsymbol{x}), H] &= \frac{-1}{2m}\nabla^2\phi(\boldsymbol{x}) + \frac{\mathrm{i}g}{m}\boldsymbol{A}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}) \\
&\quad -\frac{\mathrm{i}g^2}{2m}\int \mathrm{d}^2y\,\phi^*(\boldsymbol{y})\boldsymbol{\nabla}\times G(\boldsymbol{y}-\boldsymbol{x})\phi(\boldsymbol{x})\cdot\boldsymbol{\nabla}\phi(\boldsymbol{y}) \\
&\quad +\frac{\mathrm{i}g^2}{2m}\int \mathrm{d}^2y\,\boldsymbol{\nabla}\phi^*(\boldsymbol{y})\cdot\boldsymbol{\nabla}\times G(\boldsymbol{y}-\boldsymbol{x})\phi(\boldsymbol{x})\phi(\boldsymbol{y}) \\
&\quad +\frac{g^2}{2m}\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x})\phi(\boldsymbol{x}) + \frac{2g^3}{4\pi m}\int \mathrm{d}^2y\,\rho(\boldsymbol{y})\boldsymbol{\nabla}\times G(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{y})\phi(\boldsymbol{x}) \\
&\quad +\frac{g^4}{2m}\int \mathrm{d}^2y\,(\boldsymbol{\nabla}\times G(\boldsymbol{y}-\boldsymbol{x}))^2\rho(\boldsymbol{y})\phi(\boldsymbol{x}).
\end{aligned} \tag{25}$$

This is the same as the equation found in [23], except that we have fermionic statistics here, which as a result leads to the absence of a term resulting from the matter–magnetic field interaction. The number operator $N = \int \mathrm{d}^2x\,\rho(\boldsymbol{x})$ with $\rho(\boldsymbol{x}) = \phi^*(\boldsymbol{x})\phi(\boldsymbol{x})$ can be seen to commute with the Hamiltonian, and therefore, they can be simultaneously diagonalized. Their eigenstates are labelled as $|E, N\rangle$ [23]:

$$H|E, N\rangle = E|E, N\rangle \tag{26}$$

$$N|E, N\rangle = N|E, N\rangle. \tag{27}$$

A vacuum state $|\Omega\rangle$ is assumed to exist satisfying

$$\phi(\boldsymbol{x})|\Omega\rangle = 0 = \langle\Omega|\phi^*(\boldsymbol{x}) \tag{28}$$

$$H|\Omega\rangle = 0 = N|\Omega\rangle. \tag{29}$$

The N-particle state is defined as usual:

$$\langle\Omega|\phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\ldots\phi(\boldsymbol{x}_N)|E, N\rangle = U_E(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N). \tag{30}$$

Obviously, the state $U_E(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N)$ is a fermionic wavefunction, antisymmetric under the permutation of the positions of any two particles. The N-body Schrödinger equation can be derived by considering the commutator

$$\langle\Omega|[\phi(\boldsymbol{x}_1)\ldots\phi(\boldsymbol{x}_N), H]|E, N\rangle = EU_E(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N). \tag{31}$$

The two-body equation is the first non-trivial case of the above equation, and reads:

$$\langle\Omega|[\phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2), H]|E, 2\rangle = EU_E(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{32}$$

Evaluating the commutator, using the Heisenberg equations of motion, one obtains

$$EU_E(\boldsymbol{x}_1, \boldsymbol{x}_2) = \left[\frac{-1}{2m}\sum_{i\neq j}^{2}(\boldsymbol{\nabla}_i - \mathrm{i}g^2\boldsymbol{\nabla}_i \times G(\boldsymbol{x}_i - \boldsymbol{x}_j))^2\right]U_E(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{33}$$

Using the explicit form of the Green function (equation (20)), and adopting the centre of mass (cm) coordinates

$$\boldsymbol{R} = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2} \tag{34}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{35}$$

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 \qquad (\boldsymbol{p}_i = -\mathrm{i}\boldsymbol{\nabla}_i) \tag{36}$$

$$\boldsymbol{p} = \frac{\boldsymbol{p}_1 - \boldsymbol{p}_2}{2} \tag{37}$$

we find in polar coordinates (r, ϕ) (after separating the cm motion):

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\phi} + \frac{\mathrm{i}g^2}{2\pi}\right)^2 + k^2\right]U_k(r, \phi) = 0 \tag{38}$$

where $k^2 = m(E - \frac{K^2}{4m})$, and $\frac{K^2}{4m}$ is the energy of the cm. The above equation, which does not contain the delta function term, is formally the same as the equation studied by Aharonov and Bohm for the scattering of a charged particle off an infinitely long flux line. It also looks like the equation obtained from a scalar field theory [4, 10]. This similarity might be misleading, however, since there is a very crucial difference between this equation and the AB equation. In the AB case, the wavefunction (that is regular at the origin) was taken as a linear combination of $J_{|m+\alpha|}(kr)\mathrm{e}^{\mathrm{i}m\phi}$ ($\alpha = \frac{g^2}{2\pi}$ in our notation), with m any positive or negative integer, or zero. The wavefunction, being a single-particle wavefunction, was required only to be regular at the origin, and be single-valued. Therefore, the above choice of m as any integer, including zero, is the natural choice. In our case, however, we have a two-particle wavefunction, which needs to satisfy a further symmetry requirement resulting from the statistics of identical particles. In particular, the wavefunction $U_k(r, \phi)$ should be antisymmetric under the exchange of the coordinates of the two particles, which in polar coordinates means

$$U_k(r, \phi + \pi) = -U_k(r, \phi). \tag{39}$$

This condition, then, imposes a restriction on the values of m that enter the linear combination of the functions $J_{|m+\alpha|}(kr)\mathrm{e}^{\mathrm{i}m\phi}$, in that m is now allowed to be an odd integer only! In the case of the equation derived in [4] from a bosonic field theory, the statistical considerations force m to be an even integer (including zero). The effect of these statistical considerations appears in a most pronounced manner, when one considers the evaluation of the AB scattering amplitude perturbatively by employing the Born approximation. While a detailed field theoretic perturbative calculation of the two-particle scattering amplitude up to one-loop order is carried out in the next section, obtaining a result that coincides with the same order term in the power series expansion of the exact amplitude, a brief discussion of this issue is in order here. It was well known for many years [5, 6] that the first-order Born approximation of the AB scattering amplitude fails to give the correct result. To be more explicit, let us note that the exact AB amplitude for non-forward scattering† (for $|\alpha| < 1$) is given as

$$f(k,\phi) = -\mathrm{i}(2\pi k)^{\frac{-1}{2}} \sin\pi\alpha \left[\cot\frac{\phi}{2} - \mathrm{i}\,\mathrm{sgn}(\alpha)\right] \qquad \phi \neq 0. \tag{40}$$

Employing the first-order Born approximation to calculate the amplitude to first order in α, one, instead of obtaining

$$f(k,\phi) = -\mathrm{i}\alpha\left(\frac{\pi}{2k}\right)^{\frac{1}{2}} \left[\cot\frac{\phi}{2} - \mathrm{i}\,\mathrm{sgn}(\alpha)\right] + \mathrm{O}(\alpha^3) \qquad \phi \neq 0, \pi \tag{41}$$

obtains the incorrect result [6, 10]

$$f(k,\phi) = -\mathrm{i}\alpha\left(\frac{\pi}{2k}\right)^{\frac{1}{2}} \left[\cot\frac{\phi}{2}\right] + \mathrm{O}(\alpha^2) \qquad \phi \neq 0, \pi \tag{42}$$

in which the term non-analytic in α is missing. The failure of the Born approximation was shown [6] to be due to the fact that the first-order Born amplitude misses the contribution of the $(m = 0)$ s-partial wave which is quadratic in α. Many works aiming at solving this problem by different methods were published recently [7–10]. In particular, the works [8, 10] attempted to resolve this issue in the context of a non-relativistic scalar field theory by introducing a contact interaction term.

In our case, where we have two identical spin-$\frac{1}{2}$ fermions, the absence of the $m = 0$ partial wave in the solution of the equation (38) suggests that the Born approximation might work well. To show that this is indeed the case, first note that for identical particles, (40) is to be modified. That is, one must symmetrize (antisymmetrize) this amplitude by adding to (subtracting from) it the exchange amplitude $f(\phi - \pi)$ for the bosons (fermions) and divide by $\sqrt{2}$ to preserve normalization. In doing so, one obtains for the fermionic case

$$f(k,\phi) = -\mathrm{i}(\pi k)^{-\frac{1}{2}} \frac{\sin\pi\alpha}{\sin\phi} \qquad \phi \neq 0, \pi. \tag{43}$$

We see from (43) that the fermionic amplitude does not contain the troublesome non-analytic part in α. That the Born approximation works well in this case, is solely due to the fermionic statistics.

4. One-loop perturbative calculations

In this section, we will carry out the the calculation of the scattering amplitude of two NR fermions interactiong through the pure CS field, perturbatively up to one-loop order, starting from the theory described by the action (16), namely the local action before solving the Gauss' law constraint to obtain (22), where the gauge field is expressed in terms of the matter fields by the non-local expression (19). The motivation for doing this calculation is to support the arguments that we presented in section 3. In particular, we shall explicitly demonstrate that the diagrams coming from the Pauli term in the classical action, equation (16) play no role in rendering the two-particle scattering amplitude finite up to one-loop order, in contrast to some comments made in the literature [10] stating that this term plays the role of the contact interaction term introduced into the scalar theory.

† Generally, the unitarity of the S-matrix of the scattering requires that the amplitude includes a δ-function in the forward direction [24]. We omit this here, since we consider non-forward scattering.

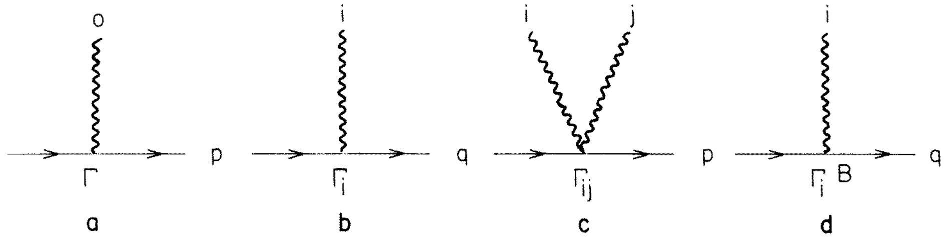

Figure 1. The vertices of the NR theory.

As our model (equation (16)) is similar, in a sense, to the scalar model considered in [10], we obtain some diagrams that are similar to some of those in this reference. However, we show that the fermionic nature of the fields in our model gives results that are dramatically different from those .

Although, as is well known, there are no real CS photons, the existence of a CS propagator enables one to formulate Feynman rules in which the gauge field propagator appears in internal lines. This can be done, for example, by noting that the S-matrix of the model is formally identical to that of models with 'real' gauge photons (the electromagnetic field, for instance). This point has been discussed thoroughly in [20, 21], and some kind of Wick-like theorem was constructed for the CS field.

To derive the Feynman rules, we first add to the action (16) the gauge fixing term

$$S = \frac{1}{\xi}\int \mathrm{d}^3x\,(\boldsymbol{\nabla}.\boldsymbol{A})^2. \tag{44}$$

Then the gauge field propagator (we choose to work in the Landau gauge, $\xi = 0$) is

$$D_{i0}(k_0,\boldsymbol{k}) = -\mathrm{i}D_{0i}(k_0,\boldsymbol{k}) = -\mathrm{i}\varepsilon_{ij}\frac{k_j}{|\boldsymbol{k}^2|} \tag{45}$$

and the matter field propagator is:

$$D(k_0,\boldsymbol{k}) = \frac{1}{k_0 - \frac{\boldsymbol{k}^2}{2m} + \mathrm{i}\epsilon}. \tag{46}$$

The vertices of the theory are (see figure 1):

$$\Gamma_0 = \mathrm{i}g \tag{47}$$

$$\Gamma_i = \frac{\mathrm{i}g}{2m}(p+q)_i \tag{48}$$

$$\Gamma_{ij} = \frac{-\mathrm{i}g^2}{m}\delta_{ij} \tag{49}$$

$$\Gamma_i^B = \frac{-g}{2m}\varepsilon_{ij}(p-q)_j. \tag{50}$$

The vertex Γ_i^B does not appear in the scalar model. There, instead, one has a vertex that comes from the contact interaction term [10].

The tree-level diagrams are shown in figure 2. In the cm frame, the amplitude of diagram (a) is:

$$A_a^{\mathrm{tree}} = \frac{-2\mathrm{i}g^2}{m}\frac{\boldsymbol{P}\wedge\boldsymbol{P}'}{(\boldsymbol{P}-\boldsymbol{P}')^2} - (\boldsymbol{P}' \to -\boldsymbol{P}')$$

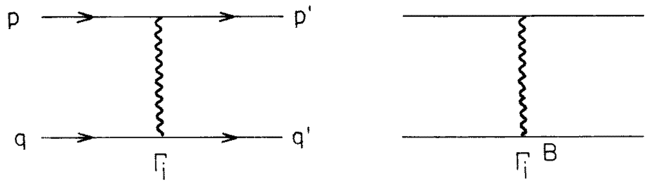

Figure 2. The tree-level diagrams.

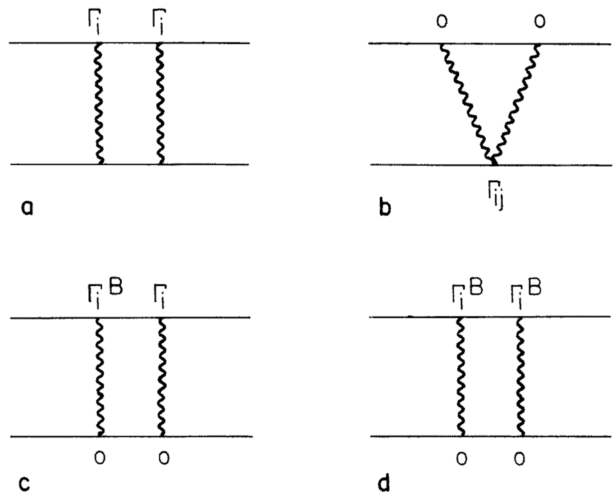

Figure 3. The one-loop diagrams.

$$
\begin{aligned}
&= \frac{-\mathrm{i}g^2}{m}\frac{\sin\theta}{(1-\cos\theta)} - (\theta \to \theta - \pi) \\
&= \frac{-\mathrm{i}g^2}{m}\sin\theta\left(\frac{1}{(1-\cos\theta)} + \frac{1}{(1+\cos\theta)}\right) \qquad (51)
\end{aligned}
$$

where $\boldsymbol{P}$ and $\boldsymbol{P}'$ are the relative incident and scattered momenta respectively, and θ is the angle between the incident and scattered momenta in the cm frame. The amplitude of diagram (b) is similarly

$$
\begin{aligned}
A_b^{\text{tree}} &= \frac{-g^2}{m}\frac{(\boldsymbol{P}-\boldsymbol{P}')^2}{(\boldsymbol{P}-\boldsymbol{P}')^2} - (\boldsymbol{P}' \to -\boldsymbol{P}') \\
&= \frac{-g^2}{m} - \left(\frac{-g^2}{m}\right) = 0. \qquad (52)
\end{aligned}
$$

In the scalar model, one has instead of diagram (b), a diagram coming from the contact interaction vertex, with a non-vanishing contribution to the scattering amplitude. Our total tree-level amplitude is therefore

$$
A_{\text{tot}}^{\text{tree}} = \frac{-\mathrm{i}g^2}{\mu m}\sin\theta\left(\frac{1}{(1-\cos\theta)} + \frac{1}{(1+\cos\theta)}\right) = \frac{-2\mathrm{i}g^2}{\mu m\sin\theta}. \qquad (53)
$$

The one-loop diagrams are depicted in figure 3.

Diagrams (a) and (b) also appear in the scalar theory, while diagrams (c) and (d), which arise from the Pauli term, do not. Instead, the scalar theory contains a diagram that comes from the contact interaction term in the Lagrangian.

Consider diagram (a) first. This diagram is finite, and after carrying out the k_0-integral we obtain

$$A_a^{(1)} = \frac{g^4}{m}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{4(\boldsymbol{k}\wedge\boldsymbol{P})(\boldsymbol{k}\wedge\boldsymbol{P}')}{(k^2-P^2-\mathrm{i}\epsilon)(\boldsymbol{P}+\boldsymbol{k})^2(\boldsymbol{P}'+\boldsymbol{k})^2} - (\boldsymbol{P}'\to-\boldsymbol{P}')\right]. \tag{54}$$

The integrand can be written in the form [10]:

$$\frac{1}{k^2-P^2-\mathrm{i}\epsilon}\left[-1+\frac{k^2+P^2}{(\boldsymbol{k}+\boldsymbol{P})^2}+\frac{k^2+P^2}{(\boldsymbol{k}+\boldsymbol{P}')^2}+\frac{4k^2\boldsymbol{P}\boldsymbol{P}'-(k^2+P^2)^2}{(\boldsymbol{k}+\boldsymbol{P})^2(\boldsymbol{k}+\boldsymbol{P}')^2}\right]-(\boldsymbol{P}'\to-\boldsymbol{P}'). \tag{55}$$

Performing the angular integration yields

$$A_a^{(1)} = \frac{g^4}{m}\int\frac{\mathrm{d}(k^2)}{4\pi}\left[\frac{1}{k^2-P^2-\mathrm{i}\epsilon}\left(-1+\frac{(k^2+P^2)|k^2-P^2|}{(k^2-P^2\mathrm{e}^{\mathrm{i}\theta})(k^2-P^2\mathrm{e}^{-\mathrm{i}\theta})}\right)-(\theta\to\theta-\pi)\right]. \tag{56}$$

Carrying out the k^2-integral, we finally obtain

$$A_a^{(1)} = \frac{g^4}{4\pi m}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right). \tag{57}$$

This result is different from the scalar case because of the antisymmetrization in the amplitude.

We next consider the triangle diagram, figure 3(b). The amplitude after performing the k_0-integral reads

$$A_b^{(1)} = \frac{-g^4}{m}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{P}+\boldsymbol{P}')}{k^2(\boldsymbol{k}-\boldsymbol{P}+\boldsymbol{P}')^2}-(\boldsymbol{P}'\to-P)'\right]. \tag{58}$$

Performing the angular integration yields

$$A_b^{(1)} = \frac{-g^4}{4\pi m}\int\mathrm{d}(k^2)\left[\frac{1}{2k^2}+\frac{1}{2|k^2-(\boldsymbol{P}-\boldsymbol{P}')^2|}-(\boldsymbol{P}'\to-\boldsymbol{P}')\right]. \tag{59}$$

In the scalar case, the above integral with the exchange term added leads to a logarithmic divergence [10], where it was regularized by the introduction of an ultraviolet cut-off. It was this divergence that made the scalar theory of this reference non-renormalizable unless a contact-interaction was introduced. In our case, however, as we are subtracting the exchange amplitude, the divergences are nicely cancelled, yielding a finite result, namely

$$A^{(1)} = \frac{-g^4}{4\pi m}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right). \tag{60}$$

The above amplitude is, not only finite, but also equals the minus of the amplitude of the box diagram, equation (57); so they cancel.

The amplitude of diagram (c) after performing the k_0-integral is:

$$A_c^{(1)} = \frac{g^4}{m}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{1}{k^2-P^2-\mathrm{i}\epsilon}-(\boldsymbol{P}'\to-\boldsymbol{P}')\right]. \tag{61}$$

This integral is similar to the one that is generated from the contact interaction term in the scalar theory (which was logarithmically divergent in that case). In our case, however, we note that this integral, being independent of the scattering angle vanishes upon subtracting the exchange amplitude. Thus

$$A_c^{(1)} = 0. \tag{62}$$

Finally, we consider diagram (d),which after carrying out the k_0-integral reads:

$$\begin{aligned}
A_d^{(1)} &= \frac{-4\mathrm{i}g^4}{m} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left[\frac{\boldsymbol{k} \wedge \boldsymbol{P}'}{(\boldsymbol{P}' + \boldsymbol{k})^2 (k^2 - P^2 - \mathrm{i}\epsilon)} - (\boldsymbol{P}' \to -\boldsymbol{P}') \right] \\
&= \frac{-16\mathrm{i}g^4}{m} \int \frac{k\,\mathrm{d}k}{(2\pi)^2} \left[\frac{1}{(k^2 - P^2 - \mathrm{i}\epsilon)} \int_0^{2\pi} \mathrm{d}\phi\, \frac{kp \sin\phi}{p^2 + k^2 + 2pk \sin\phi} \right. \\
&\qquad \left. -(\theta \to \theta - \pi) \right].
\end{aligned} \tag{63}$$

Again, noting that there is no θ-dependence in the above integral, we see that it vanishes upon subtracting the exchange term, thus

$$A_d^{(1)} = 0. \tag{64}$$

Therefore, the total one-loop amplitude vanishes

$$A_{\text{tot}}^{(1)} = A_a^{(1)} + A_b^{(1)} + A_c^{(1)} + A_d^{(1)} = 0 \tag{65}$$

and the contribution to the scattering amplitude comes only from the tree-level diagrams:

$$A(p,\theta) = A_{\text{tot}}^{\text{tree}} = \frac{-2\mathrm{i}g^2}{m \sin\theta}. \tag{66}$$

To compare our amplitude with the term of the same order in the exact total AB amplitude, equation (43), we develop the power series expansion of the latter, as

$$\begin{aligned}
f_{AB}(k,\theta) &= -\mathrm{i}(\pi k)^{\frac{-1}{2}} \frac{\sin \pi\alpha}{\sin\theta} \\
&= -\mathrm{i}(\pi k)^{\frac{-1}{2}} \frac{\pi\alpha}{\sin\theta} + \mathrm{O}(\alpha^3) \\
&= \frac{(\pi k)^{\frac{-1}{2}}}{4} \frac{-2\mathrm{i}g^2}{m \sin\theta} + \mathrm{O}(g^6) \\
&= \frac{(\pi k)^{\frac{-1}{2}}}{4} A(p,\theta) + \mathrm{O}(g^6)
\end{aligned} \tag{67}$$

where $A(p,\theta)$ is our amplitude given in (66).

So, the first-order term in the power series expansion of the AB scattering amplitude calculated from the Schrödinger equation (38) coincides—up to kinematical factors—with the result found perturbatively up to one-loop order.

Before leaving this section, we would like to emphasize again that the reason behind getting a divergence-free, finite amplitude at one-loop order (i.e. the first-order Born approximation) that is consistent with the exact amplitude, is solely the fermionic statistics of the fields. This result confirms our arguments at the end of section 3 concerning the validity of the Born approximation for fermions. In particular, we would like to stress that the Pauli term (the matter–magnetic field interaction term) in the action of the theory (equation (16)), made no contribution to the amplitude, contrary to the statement made in [10], where it was mentioned that this term will play the role of the contact interaction term in the scalar theory.

5. Conclusions

We have considered a field theory model of NR spin-$\frac{1}{2}$ fermions interacting through an Abelian CS gauge field. While the two-particle sector of the Hilbert space of the quantum field theory leads to a two-particle AB-like Schrödinger equation that is formally similar to the one resulting from similar bosonic theory, the fermionic statistics of the particles gives rise to a different two-particle scattering amplitude. It was shown that as a direct consequence of these statistical considerations, the s-partial wave does not contribute to the scattering amplitude. Therefore, the first-order Born approximation, in contrast to the case with scalar particles, works well in this case.

The classical action of our model, that is the NR limit of a relativistic fermionic action, contains a Pauli (matter–magnetic field interaction) term. In the scalar field theory models [10], the similar contact-interaction term plays a crucial role in the perturbative calculation of the two-particle scattering amplitude at one-loop order, in that it makes the amplitude renormalizable, and coincide with the AB amplitude to the same order. Therefore, we carried out a perturbative calculation of the scattering amplitude of two identical fermions up to one-loop order. We found, however, that the contribution of all the diagrams coming from the Pauli term to the scattering amplitude vanishes upon antisymmetrizing, indicating that this term plays no role in our theory. The contribution of the one-loop diagrams to the resulting scattering amplitude is finite, free of divergences and null, and that of the tree-level diagrams coincides with the exact AB amplitude to the same order. This result is solely due to the fermionic statistics of the fields, i.e to the fact that one—having identical fermions—needs to antisymmetrize the amplitude. It is also interesting to note here that the contribution of the Pauli term to the Hamiltonian (22) from which the two-particle Schrödinger equation was derived in the two-particle sector, vanished also upon normal ordering the quantized fields, due, again, to the fermionic statistics of these fields.

We have also calculated the relativistic tree-level scattering amplitude of two fermions, based on the theory given by the relativistic action (9). We would like to quote this result for the sake of completeness. In this case there is a single interaction vertex, and thus a single tree-level diagram. We have found that in the NR limit, keeping terms to $O(\frac{p}{m})$, one obtains exactly the amplitude of the NR model given in equation (66).

After this work was completed, a work by Hagen [25] was brought to our attention, which discusses the same problem from a rather more general point of view. In that work the scattering of two non-identical fermions with possible different orientations of the spins of the particles, is considered. Therefore, the Pauli term does contribute to both the Schrödinger equation and the scattering amplitude in this case. The contribution of the one-loop order diagrams to the scattering amplitude of two non-identical fermions was also shown to be finite, and null *only* for the case of parallel spins of the two particles.

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